My image of the textbook “Basic Category Theory by Tom Leinster”  
(This might contain some misunderstandings)  
by Akihiko Koga, 14th Aug. 2017

1. Basics of Categories
   - Categories
   - Functors
   - Natural Transformations

2. Adjoins
   - Adjoins
   - via Hom-sets
   - D(F(X), Y)
   - C(X, G(Y))
   - Units & Counits
   - Initial Objects

3. Interlude on sets
   - Properties of sets as building blocks
   - Small vs. Large
   - History

4. Representables
   - Yoneda lemma & its Consequences

5. Limits
   - Limits
   - Colimits
   - Interactions
     - preservation
     - creation of lim./colim.

6. Adjoins ↔ Representables ↔ Limits
   - Limits in terms of
   - Representables
   - Adjoins
   - Presheaves
   - Adjoint Functors
1. Categories, functors and natural transformations
in “Basic Category Theory” by Tom Leinster, 2014

- Category is System of related Objects

- Typical Examples

<table>
<thead>
<tr>
<th>Category</th>
<th>Objects</th>
<th>Maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grp</td>
<td>groups</td>
<td>homomorphisms</td>
</tr>
<tr>
<td>Top</td>
<td>topological spaces</td>
<td>continuous maps</td>
</tr>
</tbody>
</table>

- We will see
  - Many examples, some of them are of very **different flavour** from **those two**
  - “maps” need not be anything like maps you are familiar to. e.g., categories themselves are mathematical objects.

- Early 1940s
  - Researchers in Algebraic Topology used the term informally.
  - Samuel Eilenberg and Saunders MacLane defined **Natural Transformation** precisely. But, for the purpose, they had to define Functors and Categories, first.

  **Category Theory** began

  Now, it has spread far beyond algebraic topology

  Most parts of pure mathematics

  Applied mathematics

  Computer science
2. Adjoints

in “Basic Category Theory” by Tom Leinster, 2014

The slogan of Saundes Mac Lane's book 
Categories for Working Mathematician

![diagram]

- We will see the truth of this meeting

Examples of Adjoint functors from diverse parts of mathematics

complement from three directions with own intuitions

Isomorphism between Hom-sets

\[ B(F(A), B) \cong A(A, G(B)) \]

naturally in A and B

Units and Counits

\[ \eta : 1_A \to GF \]
\[ \varepsilon : FG \to 1_B \]

Initial object of comma category

\[ A \Rightarrow G \]

Characterizations of Adjointness which are equivalent to one another

Knowing adjointness gives you a valuable addition to your mathematical toolkit.

More should know about Adjoints. They are both common and easy. They help you to spot patterns in the mathematical landscape.

Who put these into my toolkit?

Leinster?
3. Inderlude on sets
in “Basic Category Theory” by Tom Leinster, 2014

- **Sets and Functions** are ubiquitous in mathematics from pure math to applied math.

  - For example, think of
    - probability density functions in statistics
    - data sets in experimental science
    - planetary motion in astronomy
    - flow in fluid dynamics

- **Category Theory**

  **common**
  constructions & patterns in math

- In this chapter, we will study

  - **Section 1** Common constructions & patterns on set theory
    empty set $\emptyset$, singleton set 1 ($\{x\}$), $A \times B$, $A + B$, $A^B$, $2 (= \{\text{true, false}\}$ ), Equalizers, Quotients, Natural Numbers, Choice

  - **Section 2** Small v.s. Large
    Category is a **collection** of objects and maps

    Distinction of 'collections'
    
    Small collections (set size)    Large collections (non set size)

    We use this distinction at “Adjoint Functor Theorem” in section 6.

  - **Section 3** Historical look at set theory
    It may provide useful perspective, while it is logically unnecessary for the latter chapters.

    I think this is an issue of foundation. They say there are a lot of fierce battles in the foundation.

    - You mean the foundation "the Encyclopedists" predicted to appear after the decay of Galactic Empire?

    - For a while, please forget axiomatic set theory even if you know it.

    - Maybe, no! And, only few know the old story.

    - Yes! You have read the series too?
4. Representables
in “Basic Category Theory” by Tom Leinster, 2014

- **Category** = a world of objects looking at one another

  ![Diagram](image)

  each sees the world from a different view point

- **Examples of View**

<table>
<thead>
<tr>
<th>Category</th>
<th>Object who sees</th>
<th>Object(Seen)</th>
<th>A map as a view</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topological space $\text{Top}$</td>
<td>one-point space $I$</td>
<td>a space $X$</td>
<td>a point of $X$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{R}$</td>
<td></td>
<td>a curve in $X$</td>
</tr>
<tr>
<td>Group $\text{Grp}$</td>
<td>$\mathbb{Z}$</td>
<td>a group $G$</td>
<td>an element in $G$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/p$</td>
<td></td>
<td>elements of order 1 or $p$</td>
</tr>
<tr>
<td>Field with ring morphs.</td>
<td>field $K$</td>
<td>field $L$</td>
<td>$L$ if $L$ is an extension of $K$</td>
</tr>
<tr>
<td>($\mathbb{R}, \leq$)</td>
<td>0</td>
<td>$x \in \mathbb{R}$</td>
<td>$x$ if $x \geq 0$</td>
</tr>
</tbody>
</table>

- **Dual questions**
  How an object is seen from other objects?

  Let $S$ be the set $\{0, 1\}$ and $X$ be a set. $\text{Set}(X, S)$ corresponds to the set of all subsets of $X$.

- **This chapter explores**
  Theme of
  "How each object sees and is seen by the category in which it lives"

  naturally lead to

  Representable functors which provide the second approach to the Universal Property.
5. Limits
in “Basic Category Theory” by Tom Leinster, 2014

- We have seen two approaches to Universal Property. Now, the third one.

### Universal Property

1st approach
Adjoint

\[
\begin{array}{ccc}
A & \xrightarrow{F(A)} & B \\
\text{G(B)} & \xrightarrow{A} & B
\end{array}
\]

betweeen categories

2nd approach
Representables

\[
\begin{array}{ccc}
A & \xrightarrow{F(A)} & B \\
A & \xrightarrow{B} & A(A, B)
\end{array}
\]

set-valued functors

3rd approach
Limits/Colimits

\[
\begin{array}{ccc}
\text{D}(I) & \xrightarrow{\text{Obj1}} & \text{Obj2} \\
\text{Obj3} & \xrightarrow{\text{colim}} & \text{Obj}
\end{array}
\]

Inside a category

Many familiar constructions in mathematics

- General Method
  Taking some objects and maps

\[
\begin{array}{ccc}
\text{lim.} & \text{colim.}
\end{array}
\]

New object
Limit/Colimit

- Examples
  - Category of Groups
    \[
    \begin{array}{ccc}
    \text{ker}(h) & \xrightarrow{G} & H \\
    \text{ker}(h) & \xrightarrow{G} & H
    \end{array}
    \]
    limit

  - Poset of Natural Numbers, ordered by divisibility
    \[
    \begin{array}{ccc}
    \text{N} & \xrightarrow{m} & \text{lcm}(m, n) \\
    \text{N} & \xrightarrow{n} & \text{lcm}(m, n)
    \end{array}
    \]
    colimit

(largest common multiple)

Whenever you meet a method for taking some objects and maps, and constructing a new object, it is a good chance you are looking at a limit/colimit.

So, all we have to do is only looking? Maybe. As usual

No! Move your hands actually!
6. Adjoints, representables and limits
in “Basic Category Theory” by Tom Leinster, 2014

- We have approached from three directions
  
  - Now, we work out the connections
  
  - In principle, anything that can be described in one of these formalisms can be described in the others, like the two representations of a single point

- Highlights (1) - (5)

  (1) if $B$ has all limits/colimits of shape $I$, $[A, B]$ also has all limits/colimits of shape $I$. Therefore, $[A^{op}, Set]$ has all Limits/Colimits

  From the properties (2), (3) and Yoneda, we can deduce $[A^{op}, Set]$ has all Exponentials $C^B$

  $[A^{op}, Set]$ is a Cartesian Closed Category (CCC)

  (CCC = a category with all finite products and all exponentials)

  More over, we have an important road to Topos.

(5) Topos Theory ($[A^{op}, Set]$ is a Topos)
The beginning of an incredible story that brings together the subjects of logic and geometry

(Logical Aspect)
Topos can be regarded as a universe of “sets”

(Geometric Aspect)
Topos can be regarded as a generalized topological space

- Logical Aspect
  Topos can be regarded as a universe of “sets”

- Geometric Aspect
  Topos can be regarded as a generalized topological space

By the way, one more important topic in this chapter is the adjoint functor theorems such as

"(4) A functor with a left adjoint preserves limits."

Narrow indeed, No! Dense indeed.